Some General Results for Multi-dimensional Compactons in Generalized N-dimensional KdV Equations

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Abstract

We derive a general theorem relating the energy and momentum with the velocity of any solitary wave solution of the generalized KdV equation in N-dimensions that follows from an action principle. Further, we show that our N-dimensional Lagrangian formulation leads to a subclass of the equations discussed recently by Rosenau, Hyman and Staley.

1 Introduction

Recently, in an interesting paper, Rosenau, Hyman and Staley [1] have shown the existence of spherically symmetric compactons in arbitrary number of dimensions but traveling uniformly in x direction. The purpose of this article is to show that a subclass of these compacton equations (in arbitrary number of dimensions) can be derived from a Lagrangian. For this subclass, we derive a general theorem relating the energy and momentum, with the velocity of any solitary wave solution even if the solution is not known in a closed form, and using this information comment about the stability of this subclass of compacton solutions.

Compactons were originally discovered by Rosenau and Hyman (RH) [2] in their investigation of a class of generalized KdV equations described by the parameters (m, n).

$$u_t + (u^m)_x + (u^n)_{xxx} = 0. \tag{1}$$

They found [2, 3] among other things, that these equations have solitary wave solutions of the form $A[\cos(d\xi)]^{2/(m-1)}$ for $m = 2, 3$, where $\xi = x - ct$ and $-\pi/2 \leq d\xi \leq \pi/2$. However these systems of equations were not in general derivable from a Lagrangian. For that reason, Cooper, Shepard and Sodano (CSS) [4] were led to consider a related system of equations derivable from a Lagrangian. By starting with the first order Lagrangian [4, 5]

$$\mathcal{L}(l,p) = \int \left(\frac{1}{2}\varphi_x\varphi_t + \frac{(\varphi_x)^l}{l(l-1)} - \alpha(\varphi_x)^p(\varphi_{xx})^2 \right) dx \equiv \int L dx\,, \tag{2}$$

CSS derived and studied a generalized sequence of KdV equations of the form:

$$u_t + u^{l-2}u_x + \alpha[2u^p u_{xxx} + 4pu^{p-1}u_x u_{xx} + p(p-1)u^{p-2}(u_x)^3] = 0\,, \tag{3}$$

where the usual field $u(x,t)$ of the generalized KdV equation is defined by $u(x,t) = \varphi_x(x,t)$. The equations of CSS had the same class of solutions as the original RH equations but had the advantage of being derived from an action and thus conserved energy. The CSS equations lead to similar compacton solutions when the parameters (m, n) of the RH equation have the value $(l-1, p+1)$. In general the RH and CSS equations have *different* conservation laws.

Recently, Rosenau, Hyman and Staley [1] have shown that a generalization of their original set of equations, namely

$$\mathcal{C}_{\mathcal{N}}(m, a+b) : u_t + (u^m)_x + \frac{1}{b}[u^a(\nabla^2 u^b)]_x = 0\,, \tag{4}$$

admitted traveling compacton solutions which were spherically symmetric but traveling uniformly in x direction. Here m, a and b are integers. Introducing the variables:

$$s = x - \lambda t; \quad R = \sqrt{s^2 + x_2^2 + ... + x_N^2}\,, \tag{5}$$

one can integrate their equation in a traveling frame to find

$$u^a\left[-\lambda u^{1-a} + u^{m-a} + \frac{1}{bR^{N-1}}\frac{d}{dR}R^{N-1}\frac{d}{dR}u^b\right] = 0\,. \tag{6}$$

Let us assume that the general solution of this equation is of the form

$$u = AU[\beta R]\,, \tag{7}$$

and further let the velocity dependence of A and β be

$$A \propto \lambda^\sigma\,, \quad \beta \propto \lambda^\eta\,. \tag{8}$$

Then one can immediately scale out λ provided

$$\sigma = \frac{1}{m-1}; \quad \eta = \frac{m-n}{2(m-1)}\,, \tag{9}$$

where $n = a+b$. It is amusing to note that the velocity dependence of both the amplitude A and the width β is dimension independent. Further, when $m = n$ the width is independent of the velocity of the compacton. (Note that N denotes the number of space dimensions which is different from $n = a+b$). From the RHS Eq. (4) it immediately follows that one of the conserved quantity in the RHS model is the mass M defined by

$$M = \int u d^N x\,. \tag{10}$$

Using Eqs. (8) and (9) it immediately follows that the velocity dependence of M is however dimension dependent

$$M \propto \lambda^{[N(n-m)+2]/2(m-1)}\,. \tag{11}$$

Nevertheless, for compactons since $m = n$, this N-dependence disappears even for M.

RHS found two explicit solutions in N-dimensions: (a) true compacton solution in the case $a = 1$, i.e. $\mathcal{C}_\mathcal{N}(m = 1+b,\, a+b = 1+b)$ and (b) $\mathcal{C}_\mathcal{N}(m = 2,\, n = a+b = 3)$ when the solution is part of a parabola.

We shall now show that a subclass of RHS Eq. (4) satisfying $b = a+1$ can be derived from a generalization of the CSS Lagrangian (2) with $(\varphi_{xx})^2 \to (\nabla\phi_x \cdot \nabla\phi_x)$. In particular, consider the Action

$$S = \int L(l,p)\, dxdt = \int \left(\frac{1}{2}\varphi_x\varphi_t + \frac{(\varphi_x)^l}{l(l-1)} - \alpha(\varphi_x)^p(\nabla\varphi_x \cdot \nabla\varphi_x)\right) dxdt\,. \tag{12}$$

From this we obtain the equation:

$$u_t + (\partial_x)[u^{l-1}/(l-1)] + \alpha(\partial_x)[pu^{p-1}\nabla u \cdot \nabla u] + \alpha(\partial_x)[2u^p\nabla^2 u] = 0\,. \tag{13}$$

Although this is the natural generalization of the CSS equation, at this point to make the comparison with RHS more explicit, we will modify the coefficients of the Lagrangian density to be:

$$L_2(l,p) = \frac{1}{2}\varphi_x\varphi_t + \frac{(\varphi_x)^l}{l} - \alpha(\varphi_x)^p(\nabla\varphi_x \cdot \nabla\varphi_x)\,. \tag{14}$$

The equation of motion that results from this can be written in the form:

$$u_t + (\partial_x)[u^{l-1}] + \frac{4\alpha}{(p+2)}\partial_x[u^{p/2}\nabla^2 u^{(p/2+1)}] = 0\,. \tag{15}$$

Thus by identifying

$$a = p/2\,, \quad l = m+1\,, \quad b = a+1\,, \quad \alpha = 1/2\,, \tag{16}$$

we obtain a subclass of the RHS equation with $b = a+1$. We have not found a way to generalize the CSS Lagrangian to obtain the full family of RHS equations with arbitrary b which is independent of a. For $b = a+1$, the Lagrangian (14) was also recently written down in the Appendix of [6].

2 Conservation Laws

From Eq. (15) we find that

$$M = \int u \; d^N x\,, \tag{17}$$

is conserved. This conservation law also follows from the invariance of the action under $\varphi \to \varphi + \varphi_0$, where φ_0 is a constant.

On multiplying Eq. (15) by u it is easily shown that another conserved quantity is $P = \frac{1}{2}\int u^2 \; d^N x$. In particular, the corresponding continuity equation is of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial j_k}{\partial x_k} = 0\,. \tag{18}$$

Here summation over k is understood with k counting the number of space indices. In this case the relevant matter density ρ and the current density j_k are given by

$$\begin{aligned} \rho &= \frac{u^2}{2}\,,\; j_y = \frac{2\alpha}{p+1}u^{p+1}\frac{\partial^2 u}{\partial x \partial y}\,, \quad j_z = \frac{2\alpha}{p+1}u^{p+1}\frac{\partial^2 u}{\partial x \partial z}\,, \ldots \\ j_x &= \frac{(l-1)u^l}{l} + \frac{2\alpha p}{(p+1)}u^{(p+1)}\nabla^2 u + \frac{2\alpha}{(p+1)}u^{(p+1)}\frac{\partial^2 u}{\partial x^2} + \alpha(p-1)u^p(\nabla u \cdot \nabla u)\,. \end{aligned} \tag{19}$$

A third conserved quantity is obviously the Hamiltonian corresponding to the Lagrangian density L_2 as given by Eq. (14). It is given by

$$H = \int d^N x \left[-\frac{(\varphi_x)^l}{l} + \alpha(\varphi_x)^p(\nabla\varphi_x \cdot \nabla\varphi_x) \right]\,. \tag{20}$$

More generically, the Lagrangian density L_2 in Eq. (14) is of the form:

$$L = L(\phi_\mu, \phi_{\mu\nu}). \tag{21}$$

Here we will use the compact notation that μ runs from $0, 1 \ldots N$ with $x_0 = t$ and $x_1 = x$. Also $\phi_\mu \equiv \partial\phi/\partial x_\mu$, etc. The stationary condition on the action under variations of ϕ of the generic form $\phi \to \phi + \delta\phi$, with $\delta\phi$ vanishing on the boundaries of the integration range, leads to the Euler-Lagrange equation for this type of Lagrangian:

$$\partial_\mu \frac{\delta L}{\delta \phi_\mu} = \partial_\mu \partial_\nu \frac{\delta L}{\delta \phi_{\mu\nu}}. \tag{22}$$

Here we use the Einstein summation convention that repeated indices are summed over. To obtain the energy-momentum tensor we consider an infinitesimal arbitrary coordinate transformation

$$x_\mu \to x_\mu + \epsilon_\mu(x). \tag{23}$$

Under this transformation

$$\phi(x) \to \phi(x + \epsilon(x)) \equiv \phi(x) + \delta\phi(x), \tag{24}$$

which for infinitesimal transformations leads to

$$\delta\phi(x) = \phi_\mu \epsilon_\mu. \tag{25}$$

Under this transformation we obtain:

$$\delta S = \int d^{N+1}x \left[\frac{\delta L}{\delta \phi_\mu} \partial_\mu(\phi_\lambda \epsilon_\lambda) + \frac{\delta L}{\delta \phi_{\mu\nu}} \partial_\mu \partial_\nu (\phi_\lambda \epsilon_\lambda) \right]. \tag{26}$$

One identifies the energy-momentum tensor $T_{\mu\lambda}$ from the coefficient of $\partial_\mu \epsilon_\lambda$. That is after integration by parts and using the Euler-Lagrange equation one can cast δS in the form

$$\delta S = \int d^{N+1}x \left[T_{\mu\lambda} \partial_\mu \epsilon_\lambda \right], \tag{27}$$

where

$$T_{\mu\lambda} = -\delta_{\mu\lambda} L + \frac{\delta L}{\delta \phi_\mu} \phi_\lambda + 2 \frac{\delta L}{\delta \phi_{\mu\nu}} \phi_{\nu\lambda} - \partial_\nu \left(\frac{\delta L}{\delta \phi_{\mu\nu}} \phi_\lambda \right). \tag{28}$$

The $N+1$ conservation laws that arise from the fact that

$$\partial_\mu T_{\mu\lambda} = 0 \tag{29}$$

are $dP_\mu/dt = 0$ where

$$P_\mu = \int d^N x T_{0\mu}. \tag{30}$$

The conserved energy H is P_0 and there are N conserved linear momenta P_k. In particular for the Lagrangian considered in this paper, the only fields entering the Lagrangian are ϕ_0, ϕ_x, and ϕ_{xk}. This results in the fact that

$$T_{00} = -L + \frac{1}{2} \phi_x \phi_t = \mathcal{H} \tag{31}$$

and

$$T_{0k} = \frac{1}{2}\phi_x\phi_k; \quad k = 1 \ \dots \ N. \tag{32}$$

Note that φ_k, $k \neq 1$ is only known through an integral operator, see Eq. (36) below. The tensor T_{ik} can be constructed by using

$$\delta L/\delta\varphi_i = \delta_{i1}[\varphi_t/2 + (\varphi_x)^{l-1}/(l-1) - \alpha p(\varphi_x)^{p-1}\nabla\varphi_x \cdot \nabla\varphi_c] \tag{33}$$

and

$$\delta L/\delta\varphi_{i\nu} = -2\alpha\delta_{i1}\delta_{\nu k}\varphi_{xk}(\varphi_x)^p. \tag{34}$$

Because the compactons have a preferred x direction, there is also a rotational symmetry about the x axis leading to conservation of angular momentum about that axis.

We shall see below that in general there are only two conserved quantities corresponding to the RHS Eq. (4).

3 General Properties of the Exact Solutions

Let us assume that the general solution to the field Eq. (15) is of the form

$$u \equiv \varphi_x = AZ(\beta R)\,, \tag{35}$$

where A and β are constants, Z is an unspecified function and

$$R = \left[((x + q(t))^2 + \sum_{i=2}^{N} x_k^2\right]^{1/2} . \tag{36}$$

Using the fact that β must be chosen to minimize the action we shall deduce the general behavior of the conserved quantities M, P and H as well as the amplitude A and the width β as a function of the velocity $\dot{q}$.

To evaluate the action we must first determine φ. A convenient choice for the lower value limit of integration (see also [7]) leads to:

$$\varphi(x, x_i, t) = \int_{-q(t)}^{x} dx' \, [AZ(\beta R(x' + q(t), x_i)] \ . \tag{37}$$

Introducing the rescaled variables $y' = \beta(x' + q(t))$, $y_i = \beta x_i$, we have

$$\varphi(x, t) = \frac{A}{\beta} \int_0^{\beta(x+q(t))} \ dy \ Z[(y^2 + \sum y_i^2)^{1/2}]. \tag{38}$$

Thus we find that

$$\varphi_t = \dot{q} \ \varphi_x\,, \tag{39}$$

and the the kinetic term in the action is

$$KE = \int d^N x \frac{1}{2} \varphi_x \varphi_t = \dot{q} \int d^N x \frac{1}{2} (\varphi_x)^2 = \dot{q} P \,. \tag{40}$$

Thus the effective Lagrangian has the canonical form for point mechanics:

$$L = P\dot{q} - H \,, \tag{41}$$

where

$$H = \int d^N x \, \mathcal{H} \,, \tag{42}$$

is evaluated for the field ansatz.

We can now eliminate A in all the conservation equations in favor of the conserved momentum P. Defining the constants C_l via

$$C_l = \int dr \quad Z^l(r) r^{N-1} \Omega_N \,, \tag{43}$$

where Ω_N is the solid angle in N dimensions, we find that

$$P = \frac{A^2}{2\beta^N} C_2 \,, \tag{44}$$

so that

$$A = \sqrt{\frac{2}{C_2}} \beta^{N/2} P^{1/2} \,. \tag{45}$$

The constant of motion M is given by

$$M = \int d^N x \ u = \sqrt{\frac{2}{C_2}} \ C_1 \ \frac{P^{1/2}}{\beta^{N/2}} \,. \tag{46}$$

The key point to note is that β can be determined in terms of P using the fact that the actual solution of Hamilton's equations satisfies

$$\frac{\partial H}{\partial \beta} = 0 \,. \tag{47}$$

This is a consequence of the fact that if we consider a reduced class of trial functions parametrized by β then the solution that minimizes the energy satifies Eq. (47). When the actual solution is in the class of trial functions, then the value of β that leads to a solution of the actual equations of motion is the value of β obtained from the minimization of the energy. This is due to the fact that the exact equations of motion minimize the Action.

For the Hamiltonian we find:

$$H = -f_1(l) P_x^{l/2} \beta^{N(l-2)/2} + f_2(p) P^{(p+2)/2} \beta^{(Np+4)/2} \,, \tag{48}$$

where

$$f_1(l) = \frac{C_l}{l}\left(\frac{2}{C_2}\right)^{l/2}, \qquad f_2(p) = \alpha D_p \left(\frac{2}{C_2}\right)^{(p+2)/2}, \tag{49}$$

and

$$D_p = \int dr\, r^{N-1}[Z(r)]^p \,(dZ/dr)^2\, \Omega_N \,. \tag{50}$$

Minimizing H with respect to β we obtain:

$$\beta = K(N,p,l)^{2/[N(l-p-2)-4]} P^{(p-l+2)/[N(l-p-2)-4]}, \tag{51}$$

where

$$K(N,p,l) = \frac{f_2(p)(Np+4)}{N(l-2)f_1(l)}\,. \tag{52}$$

Therefore

$$H = \frac{N(l-p-2)-4}{Np+4} f_1(l) K^{N(l-2)/[N(l-p-2)-4]} P^r, \tag{53}$$

where

$$r = \frac{(p+2)N - l(N-2)}{N(p+2-l)+4}\,. \tag{54}$$

Note that for $N = 1$ we get our previous result that

$$r = \frac{p+2+l}{6+p-l}\,. \tag{55}$$

Using

$$\lambda \equiv \dot{q} = \frac{\partial H}{\partial P}, \tag{56}$$

then we find, in analogy with the CSS equation, that we have

$$\lambda = r\frac{H}{P}, \tag{57}$$

where r is now dependent on N and given by Eq. (55). We also find that

$$P \propto \lambda^{\,[N(p+2-l)+4]/[2(l-2)]} \tag{58}$$

and

$$\beta \propto \lambda^{\,(l-p-2)/[2(l-2)]}, \quad A \propto \lambda^{\,1/[(l-2)]}\,. \tag{59}$$

In addition, we find that

$$M \propto \lambda^{\,[N(p+2-l)+2]/[2(l-2)]}\,. \tag{60}$$

It is worth noting that while the dependence of M and P on λ (i.e., $\dot{q}$) is dimension-dependent, this is not so for A and β. Note however that in the special case of compactons since $l = p+2$ hence, in that case, the dependence of H, P and M on λ is also independent of the number of dimensions N. On using the correspondence $l = m+1$, $p = n-1 = a+b-1$ between RHS and our model, it is easy to check that, as expected, the λ dependence of A, β and M is exactly as given by Eqs. (9) and (11). Note however that without knowing the Lagrangian,

one cannot say anything about the velocity dependence of energy H and momentum P which is required to study the question about the stability of such solutions.

Stability of Solutions: Following the analysis in [8], it is clear that the criterion for linear stability is equivalent to the condition

$$\frac{\partial P}{\partial \lambda} > 0\,. \tag{61}$$

Using Eq. (58), it then follows that the compacton solutions of our model (i.e. RHS model with $b = a+1$) in N-dimensions are stable provided $2 < l < p+2+4/N$. However, it is not possible to say anything about the other compacton solutions of the RHS model.

4 Comparison and Contrast Between General RHS Equation and the Lagrangian Subset

Before finishing this article, we would like to make several remarks about the RHS model [1] *vis a vis* the subset of models defined by our Lagrangian.

1. While there are $N+2$ conserved quantities (excluding angular momentum about the x axis) in our model with $b = a+1$, (i.e. H, $\int u d^N x$ and $\int u^2 d^N x$), there are in general only two known conserved quantities (i.e. $\int u d^N x$ and $\int u^{b-a+1} d^N x$) in the RHS model. While the conservation of $\int u d^N x$ is rather obvious from the RHS field Eq. (4), the conservation of $\int u^{b-a+1} d^N x$ is not so obvious. However, one can show that the continuity equation (18) is satisfied in that case with

$$\begin{aligned}
\rho &= \frac{u^{b+1-a}}{b+1-a}\,,\;\; j_y = \frac{b-a}{2b-1} u^{2b-1} \frac{\partial^2 u}{\partial x \partial y}\,,\quad j_z = \frac{b-a}{2b-1} u^{2b-1} \frac{\partial^2 u}{\partial x \partial z}\,, \ldots \\
j_x &= \frac{m u^{(m+b-a)}}{(m+b-a)} + \left(1 - \frac{a}{b}\right) \left[\frac{b}{(2b-1)} u^{(2b-1)} \frac{\partial^2 u}{\partial x^2} \right. \\
&\left. + \frac{(a+b-2)}{2} u^{(2b-2)} (\nabla u \cdot \nabla u) + \frac{(a+b-1)}{(2b-1)} u^{(2b-1)} \nabla^2 u \right].
\end{aligned} \tag{62}$$

   In the special case when $b = a+1$, then as expected this expression reduces to the one given by Eq. (19) provided we make use of the identification as given by Eq. (16).

   It may be noted that in case $b - a + 1 = 0$ then one can show in the RHS model that a third conserved quantity is $\ln u$. In particular, in that case the continuity Eq. (18) is satisfied with

$$\begin{aligned}
&\rho = \ln u\,,\;\; j_y = u^{(2b-2)} \frac{\partial u}{\partial x} \frac{\partial u}{\partial y}\,,\;\; j_z = u^{(2b-2)} \frac{\partial u}{\partial x} \frac{\partial u}{\partial z}\,, \ldots \\
&j_x = \frac{m u^{(m-1)}}{(m-1)} + u^{(2b-1)} \nabla^2 u + u^{(2b-2)} \left(\frac{\partial u}{\partial x}\right)^2 + \frac{(2b-3)}{2} u^{(2b-2)} (\nabla u \cdot \nabla u)\,.
\end{aligned} \tag{63}$$

Similarly, when $b+1-a=1$, one can show that a third conserved quantity is $u\ln u$. In particular, continuity Eq. (18) is satisfied with

$$\rho = u\ln u\,,\ j_y = -u^{(2b-2)}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\,,\ j_z = -u^{(2b-2)}\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\,,...$$
$$j_x = u^m\ln u + \frac{(m-1)u^m}{m} + (\ln u+1)u^{(2b-1)}\nabla^2 u$$
$$-u^{(2b-2)}\left(\frac{\partial u}{\partial x}\right)^2 + \left[\frac{(2b-1)}{2} + (b-1)\ln u\right]u^{(2b-2)}(\nabla u\cdot\nabla u)\,. \tag{64}$$

2. In the special case when $a=0$ and $b=m$, one can easily show that the RHS model then has two more conserved quantities given by $\int u\cos(\sqrt{b}x)d^Nx$ and $\int u\sin(\sqrt{b}x)d^Nx$. In particular, the corresponding densities are

$$\rho = u\cos(\sqrt{b}u)\,,\ j_y = bu^{(b-1)}\frac{\partial u}{\partial y}\sin(\sqrt{b}x)\,,\ j_z = bu^{(b-1)}\frac{\partial u}{\partial z}\sin(\sqrt{b}x)\,,...$$
$$j_x = -u^b\cos(\sqrt{b}x) + \frac{1}{b}\cos(\sqrt{b}x)\nabla^2 u^b + \sqrt{b}u^{(b-1)}\frac{\partial u}{\partial x}\sin(\sqrt{b}x)\,, \tag{65}$$

$$\rho = u\sin(\sqrt{b}u)\,,\ j_y = bu^{(b-1)}\frac{\partial u}{\partial y}\cos(\sqrt{b}x)\,,\ j_z = bu^{(b-1)}\frac{\partial u}{\partial z}\cos(\sqrt{b}x)\,,...$$
$$j_x = -u^b\sin(\sqrt{b}x) + \frac{1}{b}\sin(\sqrt{b}x)\nabla^2 u^b - \sqrt{b}u^{(b-1)}\frac{\partial u}{\partial x}\cos(\sqrt{b}x)\,. \tag{66}$$

We believe that the existence of these two extra conserved quantities could perhaps explain the numerical observation of RHS as to why $a=0$, $m=n$ compactons are much closer to being elastic compared to those compactons with $m=n$ but $a\neq 0$.

3. In the case of the RHS model, the compacton solutions are easily written down in N-dimensions in case $a=1$, $m=n=b+1$. In particular, following RHS [1], if we write down the solution as

$$u^b = \lambda\left[1-\frac{F(R)}{F(R_*)}\right]\,,\quad 0<R<R_*\,, \tag{67}$$

and vanishing elsewhere, then one can show that the solution in N-dimensions is

$$F(R) = R^{-k}J_k(\sqrt{b}R)\,,\ k=(N-2)/2=0,1,2,...\,,$$
$$F(R) = \left[\frac{1}{R}\frac{d}{dR}\right]^k[c_1\sin(\sqrt{b}R)+c_2\cos(\sqrt{b}R]\,,\quad k=(N-3)/2=0,1,2,... \tag{68}$$

where $J_n(x)$ is the Bessel function of order n [9].

4. Similarly, it is easy to generalize the parabolic compacton solution of RHS in the case $m=2$, $n=a+b=3$. In particular, it is easily shown that in general, if $m=(n+1)/2$ then the parabolic compacton solution to Eq. (4) is given by

$$u^{(n-1)/2} = A-BR^2\,,\quad 0<R\le R_*\equiv\sqrt{A/B}, \tag{69}$$

where

$$A = \frac{\lambda[N(m-1)+b+1-a]}{b+1-a}\,, \quad B = \frac{(m-1)^2}{2b[N(m-1)+b+1-a]}\,. \tag{70}$$

5. Apart from these two compacton solutions, it is also possible to obtain other compacton solutions to RHS Eq. (4). For example, on using the ansatz

$$s = x - \lambda t + \alpha_2 y + \alpha_3 z + \ldots\,, \tag{71}$$

in Eq. (4), it is easily shown that the RHS Eq. (4) essentially reduces to a one-dimensional problem. In particular, on integrating twice, and assuming the two constants of integration to be zero, one can show that the RHS Eq. (4) takes the form

$$\left(\frac{du}{ds}\right)^2 = \frac{2\lambda u^{3-n}}{b(b+1-a)C} - \frac{2u^{m+2-n}}{b(m+b-a)C}\,, \tag{72}$$

where $C = 1 + \alpha_2^2 + \alpha_3^2 + \ldots.$ This is a one-dimensional equation which represents the generalization of the old RH equation [2]. Following our earlier work [10], one can immediately write down the compacton as well as the elliptic and even general hyper-elliptic compacton solutions of Eq. (72). For example, for $m = n$, the compacton solution is given by

$$u = A[\cos(\beta s)]^{2/(n-1)}\,, \quad -\pi/2 \le \beta s \le \pi/2\,, \tag{73}$$

and zero elsewhere, where

$$A = \left[\frac{2b\lambda}{b+1-a}\right]^{1/(n-1)}\,,\ \beta = \frac{(n-1)}{2b\sqrt{C}} \tag{74}$$

On the other hand, for $m = 2n-1$, the elliptic compacton solution is given by

$$u = A[cn(\beta s, k^2 = 1/2)]^{2/(n-1)}\,, \quad -K(k^2 = 1/2) \le \beta s \le K(k^2 = 1/2)\,, \tag{75}$$

and zero elsewhere, where

$$A = \left[\frac{(3b+a-1)\lambda}{b+1-a}\right]^{1/2(n-1)}\,,\ \beta = \frac{(n-1)}{\sqrt{bC}}\left[\frac{\lambda}{(b+1-a)(3b+a-1)}\right]^{1/4}\,. \tag{76}$$

Here $cn(x,k)$ denotes a Jacobi elliptic function with modulus k and $K(k)$ is the complete elliptic integral of the first kind [9]. Similarly, the other solution of [10] (when $m = 3n-2$) can be written down as well.

5 Conclusion

In this article we have generalized the Lagrangian of CSS [4] to N dimensions and have obtained a system of KdV equations which are a subset of the recently discovered equations of RHS [1] which support N-dimensional compactons. By exploiting the Lagrangian structure we were able to obtain certain general relations among the conserved quantities (derived from an energy-momentum tensor) and how these quantities depend on the velocity of a traveling wave solution. Using this information we remarked on the question of the stability of such solutions, specifically one would expect the $a = 0$ compactons to be more stable due to the existence of two more $(N + 2)$ conserved quantities.

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